THERMOCAPILLARY CONVECTION IN A THIN LIQUID LAYER LOCALLY HEATED FROM ABOVE

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The phenomenon of thermocapillary convection was considered in [1] using the example of motion of a thin liquid layer in the central portion of a planar cuvette, the opposite walls of which were maintained at different temperatures, with the assumption of constant layer thickness. The heat-transfer equation in the liquid was not used, and a linear temperature distribution was postulated on the free surface. This inconsistency was noted in [2], which solved the analogous problem of motion of a liquid under the action of a surfactant additive. The effects of gravity and deformation of the liquid-free surface in the problem of [1] were considered in [3]. Finally, [4] found an exact solution of the free convection equations for a planar liquid layer with constant temperature gradient along the column. As in [5], it was found that in sufficiently thin liquid layers (less than 1 cm) the thermocapillary convection mechanism predominates. The flow velocity profile for the case of thermocapillary motion coincides with that obtained in [1]. Physically, the solution of [4] corresponds to the unrealistic case of a liquid layer heated from below in a manner such that with an inhomogeneous temperature distribution on the layer boundaries the heat flux through the boundaries is homogeneous.

The goal of the present study is an examination of steady-state two-dimensional thermocapillary motion which arises in a thin liquid layer due to localized heating from above. In this case convection due to Archimedean forces may be neglected. The liquid velocity and temperature fields are determined with the assumption that the layer thickness h is much less than the characteristic longitudinal flow dimension l, making it possible to use the approximations of boundary-layer theory. In the first approximation we will also neglect deformation of the liquid-free surface. The liquid layer bounded by the free surface ($y = 0$) and the solid wall ($y = -h$) locally heated from the direction of the free surface. Directing the x axis opposite to the temperature gradient along the layer, we will consider convection in the region $x > 0$, removed from the heating point by a distance $\gtrsim h$, where the effect of the flow rotation zone and the details of the mechanism by which heating is accomplished have no effect. Thus, at $x \geq 0$ the condition $\partial T/\partial y = 0$ on the free surface and the temperature difference $\Delta T = T_0 - T_w$ across the layer in the initial section $x = 0$ are specified. The temperature of the cuvette bottom T_w is assumed constant, which corresponds to contact of the liquid with a good heat conductor of large dimensions or contact between liquid and solid phases of the same material. In the case considered, the equations of steady-state free convection and the boundary conditions have the form

$$\partial u/\partial x + \partial v/\partial y = 0,\ \partial p/\partial x = \eta \partial^2 u/\partial y^2,\ \partial p/\partial y = 0, \qquad (1)$$
$$u\partial T/\partial x + v\partial T/\partial y = \chi \partial^2 T/\partial y^2;$$

$$T|_{y=-h} = T_w,\ \partial T/\partial y|_{y=0} = 0,\ \eta \partial u/\partial y|_{y=0} = d\sigma/dx = -\alpha \partial T/\partial x|_{y=0}, \qquad (2)$$
$$u|_{y=-h} = v|_{y=-h} = 0,\ v|_{y=0} = 0,$$

where σ is the surface tension coefficient; $\alpha = -d\sigma/dT = \text{const}$. Inertial terms have been omitted from the equation of motion in Eq. (1). The conditions under which they may be neglected will be indicated after a solution is obtained, since no characteristic velocity figures in the problem under consideration. We will limit our examination to the case in which the total liquid flux in any section is equal to zero

$$\int_{-h}^{0} u dy = 0.$$

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Integrating the equations of motion and continuity, we find

$$u = -\frac{\alpha}{4h\eta}\left(\frac{\partial T}{\partial x}\right)_{y=0}(3y^2 + 4hy + h^2),\ v = \frac{\alpha}{4h\eta}\left(\frac{\partial^2 T}{\partial x^2}\right)_{y=0} y\,(y+h)^2. \tag{3}$$

Substituting Eq. (3) in the thermal conductivity equation and transforming to dimensionless variables

$$\theta = (T - T_w)/(T_0 - T_w),\ x/h = x',\ y/h = y',$$

we obtain (primes on the coordinates are omitted)

$$-G'(y)\left(\frac{\partial\theta}{\partial x}\right)_{y=0}\frac{\partial\theta}{\partial x} + G(y)\left(\frac{\partial^2\theta}{\partial x^2}\right)_{y=0}\frac{\partial\theta}{\partial x} = \frac{4}{3\mathrm{M}}\frac{\partial^2\theta}{\partial y^2}, \tag{4}$$

where $G(y) = {}^1/_3 y(1+y)^2$, $M = h\alpha\Delta T/\eta\chi$; M is the Marangoni number. In Eq. (4) the separation of variables $\theta = XY$ may be carried out if

$$X'^2 = k^2 X, \tag{5}$$

where k is a nonzero separation constant. The solutions of Eq. (5) have the form

$$X = (1 \pm kx/2)^2, \tag{6}$$

in which for liquid temperature decreasing with increase in x the minus sign must be chosen. Thus, to determine the function Y, we arrive at the spectral problem

$$Y'' - \lambda G Y' + 2\lambda G' Y = 0,\ Y(-1) = 0,\ Y'(0) = 0 \tag{7}$$

with parameter $\lambda = 3k^2M/8$. For the case under consideration, it is sufficient to limit ourselves to the smallest positive eigenvalue λ_0, since the corresponding eigenfunction Y_0 has no nodes within the liquid layer (states corresponding to subsequent eigennumbers will not be considered). We will use the Galerkin method. Equation (7) may be written in operator form

$$\hat{M}Y = \lambda\hat{N}Y.$$

The operator $\hat{N}$ is not self-conjugate and positively defined. Therefore, according to [6], one should employ approximations of the functions w_i, for which

$$\int_{-1}^{0} w_i\hat{N}w_i\,dy > 0,\ \int_{-1}^{0} w_i\hat{N}w_k\,dy \geqslant 0 \quad (i \neq k). \tag{8}$$

Without attempting an exact calculation of the first eigenvalue and eigenfunction, we will limit ourselves to a binomial approximate solution

$$Y = C_1 w_1 + C_2 w_2, \tag{9}$$

where $w_1 = 1 - y^2$, $w_2 = 1 + y^3$ satisfy the boundary conditions (2) and conditions (8). Substituting Eq. (9) in Eq. (7) and commencing from the requirement of orthogonality of the equation residue to the approximating functions, we arrive at a system of linear Galerkin equations for the coefficients C_1 and C_2. Solution of the characteristic equation gives $\lambda_0 \simeq 5.58$. The corresponding eigenfunction satisfying the condition $y(0) = 1$ has the form

$$Y \simeq 1 - 0{,}556y^2 + 0.444y^3.$$

The solution obtained may be written in the form

$$\theta = \left(1 - \frac{x}{l}\right)^2 Y\left(\frac{y}{h}\right),\ u = 3v_0\left(1 - \frac{x}{l}\right)G'\left(\frac{y}{h}\right),\ v = 3v_0\frac{h}{l}G\left(\frac{y}{h}\right), \tag{10}$$

where

$$l = \sqrt{\frac{3}{2\lambda_0}}\,h^{3/2}\left(\frac{\alpha\Delta T}{\eta\chi}\right)^{1/2} \simeq 0{,}52h\sqrt{\mathrm{M}},\ v_0 = \sqrt{\frac{\alpha\Delta T\chi}{\eta h}}. \tag{11}$$

Thus, the convection has a cellular structure with longitudinal period l. At $x = l$ the heat flux disappears, so the characteristic scale l is the length to which heat propagates from the source. If a wall is placed in the section $x = l$ and maintained at an unperturbed temperature T_W, then the solution will describe thermocapillary motion in a cuvette heated from the other end.

In the segment $0 < x < 2l$ Eq. (10) describes convection in a cuvette symmetrically heated from above at both ends. In this case, two convective cells are formed in the liquid, with the liquid velocity at their common boundary $x = l$ being directed downward. Finally, if for negative x we use the second solution of Eq. (6), the solution on the interval $-l < x < l$ will describe liquid motion in a cuvette outside the zone at which heat is applied in the center. In this case two convective cells with descending liquid flow at the heating point are also formed. The character of the liquid motion remains as before, if h varies only slightly with coordinate x. Thus it is not difficult to sketch out a convection pattern for more general nonsymmetric cases. For example, in the case of a liquid layer of great length one can imagine a combined convective cell $2l$ in length, in the limits of which temperature perturbation is localized and a flow of the form of Eq. (10) develops. The treatment of Eq. (11) relating h and l can also be generalized in the following fashion. If the liquid layer is sufficiently deep and the size of the zone l which is perturbed by local heating of the free surface is known, then, according to Eq. (11), we may determine the thickness of the layer which is set into motion. The condition $h \ll l$ or $M \gg 1$ is valid over a wide parameter range. For example, for water and ethanol it is necessary that $h\Delta T \gg 10^{-4}$, while for mercury $h\Delta T \gg 10^{-2}$.

One of the main goals of this study is the determination of the flow velocity. The maximum velocity v_0 is reached on the surface and, according to Eq. (11), will depend on all thermophysical characteristics of the liquid. In [1, 3, 4] the expression

$$v_0 \sim \frac{h\alpha}{\eta}\frac{dT}{dx}, \tag{12}$$

was obtained, with the temperature gradient being specified equal to $(T_1 - T_2)/L$, where L is the channel length. According to Eq. (12), the motion velocity is independent of thermal conductivity coefficient and increases with increase in h. The last statement [1] contradicts the physical pattern of the phenomenon. In fact, the source of the motion is tangent stress on the free surface, produced by a gradient in the surface tension coefficient. If this quantity is fixed, then the thinner the liquid layer induced into motion, the greater will be its velocity. Apparently this contradiction can be explained by the fact that if convective heat transfer is significant solutions of the thermal conductivity equation with $\partial T/\partial x = \text{const}$ at $y = 0$ do not in general exist in the liquid. The study of the question set forth in the works cited above is not complete, and must be complemented by determining the temperature gradient for each concrete case. A correct answer can be obtained for the present case if in Eq. (12) we substituted $(dT/dx) = \Delta T/l$ and use the relationship between h and l given by Eq. (11). The velocity increases with decrease in h and increase in χ. The χ-dependence is explained by the fact that with increase in thermal conductivity l decreases and the temperature gradient on the surface increases. The flow lines (in dimensionless variables x', y') are given by the equation

$$|G(y')|(1 - \alpha_1 x') = C,$$

where $\alpha_1 = h/l$. They are shown in Fig. 1 for $\alpha_1 = 0.1$. Figure 2 shows the isotherms (numbers on the curves are θ values).

In conclusion, we note that if v_0 is known, an expression can be developed for the similarity criteria. For example, for the Peclet number we have

$$\mathrm{Pe} = v_0 2l/\chi = (\alpha \Delta T 2l/\eta\chi)^{2/3} = \mathrm{M},$$

i.e., the Peclet and Marangoni numbers coincide for this problem. Then Eq. (11) relating h and l may be written in a form similar to the expression for boundary-layer thickness on a plate

$$h = 2l/\sqrt{\mathrm{PrRe}},$$

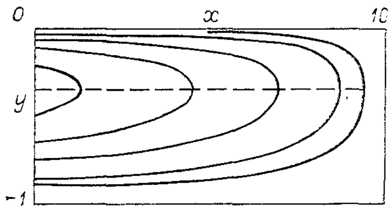

Fig. 1

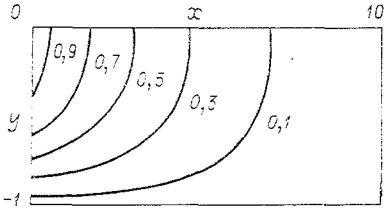

Fig. 2

where Pr = ν/χ is the Prandtl number and Re = $v_0 2l/\nu$ is the Reynolds number. Thus, the case of convection considered corresponds to motion with large Peclet numbers. We will now evaluate the conditions for applicability of this treatment more accurately. Using Eq. (10), it can be shown that the condition

$$|u\partial u/\partial x| \ll \nu|\partial^2 u/\partial y^2|$$

can be satisfied if Pr >> $^1/_6$. This inequality is satisfied by the majority of droplet-forming liquids. The theory is inapplicable to liquid metals which have Pr << 1. In that case, inertia of the liquid has an important role.

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